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# 1.4 Exponential Functions

In Appendix G we present an alternative approach to the exponential and logarithmic functions using integral calculus.

The function is called an exponential function because the variable, x, is the exponent. It should not be confused with the power functionin which the variable is the base.

In general, an exponential function is a function of the form



Where b is a positive constant. Let’s recall what this means.

If x = n, a positive integer, then



If , where n is a positive integer, then



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FIGURE 1 Representation of  , x rational

If x is a rational number,  , where p and q are integers and q > 0, then



But what is the meaning of if x is an irrational number? For instance, what is meant by?

To help us answer this question we first look at the graph of the function, where x is rational. A representation of this graph is shown in Figure 1. We want to enlarge the domain of y = 2x to include both rational and irrational numbers.

There are holes in the graph in Figure 1 corresponding to irrational values of x. We want to fill in the holes by defining, so that f is an increasing function. In particular, since the irrational number  satisfies



we must have



and we know what mean because 1.7 and 1.8 are rational numbers. Similarly, if we use better approximations for  , we obtain better approximations for :









A proof of this fact is given in J. Marsden and A. Weinstein, Calculus Unlimited (Menlo Park, CA: Benjamin/Cummings, 1981). . . . .

It can be shown that there is exactly one number that is greater than all of the numbers



and less than all of the numbers



We define to be this number. Using the preceding approximation process we can compute it correct to six decimal places:



Similarly, we can define (or  , if b > 0) where x is any irrational number. Figure 2 shows how all the holes in Figure 1 have been filled to complete the graph of the function 

FIGURE 2, x real

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The graphs of members of the family of functions  are shown in Figure 3 for various values of the base b. Notice that all of these graphs pass through the same point (0, 1) because. Notice also that as the base b gets larger, the exponential function grows more rapidly (for x > 0).

If 0 < b < 1, then approaches 0 as x becomes large. If b > 1, thenapproaches 0 as x decreases through negative values. In both cases the x-axis is a horizontal asymptote. These matters are discussed in Section 2.6.

FIGURE 3

You can see from Figure 3 that there are basically three kinds of exponential functions . If 0 < b < 1, the exponential function decreases; if b = 1, it is a constant; and if b > 1, it increases. These three cases are illustrated in Figure 4. Observe that if , then the exponential function  has domain and range  . Notice also that, since, the graph of is just the reflection of the graph of  about the y-axis.

FIGURE 4

1. 
2. 
3. 

One reason for the importance of the exponential function lies in the following properties. If x and y are rational numbers, then these laws are well known from elementary algebra. It can be proved that they remain true for arbitrary real numbers x and y.

Laws of Exponents If a and b are positive numbers and x and y are any real numbers, then

1. 
2. 
3. 
4. 

For review and practice using the Laws of Exponents, click on Review of Algebra.

## EXAMPLE 1

Sketch the graph of the function and determine its domain and range.

SOLUTION First we reflect the graph of [shown in Figures 2 and 5(a)] about the x-axis to get the graph ofin Figure 5(b). Then we shift the graph of 

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upward 3 units to obtain the graph of in Figure 5(c). The domain is and the range is.

FIGURE 5

1. 
2. 
3. 

## EXAMPLE 2

Use a graphing device to compare the exponential function and the power function. Which function grows more quickly when x is large?

Example 2 shows that increases more quickly than. To demonstrate just how quickly increases, let’s perform the following thought experiment. Suppose we start with a piece of paper a thousandth of an inch thick and we fold it in half 50 times. Each time we fold the paper in half, the thickness of the paper doubles, so the thickness of the resulting paper would be inches. How thick do you think that is? It works out to be more than 17 million miles!

SOLUTION Figure 6 shows both functions graphed in the viewing rectangle. We see that the graphs intersect three times, but for x > 4 the graph of stays above the graph of. Figure 7 gives a more global view and shows that for large values of x, the exponential function grows far more rapidly than the power function.

FIGURE 6

FIGURE 7

### Applications of Exponential Functions

The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of population growth and radioactive decay. In later chapters we will pursue these and other applications in greater detail.

First we consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the number of bacteria at time t is, where t is measured in hours, and the initial population is, then we have



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FIGURE 8: Scatter plot for world population growth

Table 1

|  |  |
| --- | --- |
| T( years since 1900) | Population (millions) |
| 0 | 1650 |
| 10 | 1750 |
| 20 | 1860 |
| 30 | 2070 |
| 40 | 2300 |
| 50 | 2560 |
| 60 | 3040 |
| 70 | 3710 |
| 80 | 4450 |
| 90 | 5280 |
| 100 | 6080 |
| 110 | 6870 |

It seems from this pattern that, in general,



This population function is a constant multiple of the exponential function, so it exhibits the rapid growth that we observed in Figures 2 and 7. Under ideal conditions (unlimited space and nutrition and absence of disease) this exponential growth is typical of what actually occurs in nature.

What about the human population? Table 1 shows data for the population of the world in the 20th century and Figure 8 shows the corresponding scatter plot.

The pattern of the data points in Figure 8 suggests exponential growth, so we use a graph- ing calculator with exponential regression capability to apply the method of least squares and obtain the exponential model



where t = 0 corresponds to 1900. Figure 9 shows the graph of this exponential function together with the original data points. We see that the exponential curve fits the data reasonably well. The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.

FIGURE 9: Exponential model for population growth

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In 1995 a paper appeared detailing the effect of the protease inhibitor ABT-538 on the human immunodeficiency virusTable 2 shows values of the plasma viral load of patient 303, measured in RNA copies per mL, t days after ABT-538 treatment was begun. The corresponding scatter plot is shown in Figure 10.

Table 2

|  |  |
| --- | --- |
| T (days) |  |
| 1 | 76.0 |
| 4 | 53.0 |
| 8 | 18.0 |
| 11 | 9.4 |
| 15 | 5.2 |
| 22 | 3.6 |

FIGURE 10 Plasma viral load in patient 303

The rather dramatic decline of the viral load that we see in Figure 10 reminds us of the graphs of the exponential function in Figures 3 and 4(a) for the case where the base b is less than 1. So let’s model the functionby an exponential function. Using a graphing calculator or computer to fit the data in Table 2 with an exponential function of the form, we obtain the model



In Figure 11 we graph this exponential function with the data points and see that the model represents the viral load reasonably well for the first month of treatment.

FIGURE 11: Exponential model for viral load

We could use the graph in Figure 11 to estimate the half-life of V, that is, the time required for the viral load to be reduced to half its initial value (see Exercise 33). In the next example we are given the half-life of a radioactive element and asked to find the mass of a sample at any time.

## EXAMPLE 3

The half-life of strontium-90,  is 25 years. This means that half of any given quantity of  will disintegrate in 25 years.

1. If a sample of  has a mass of 24 mg, find an expression for the mass that remains after t years.
2. Find the mass remaining after 40 years, correct to the nearest milligram.
3. Use a graphing device to graph and use the graph to estimate the time required for the mass to be reduced to 5 mg.

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FIGURE 12: 

SOLUTION

1. The mass is initially 24 mg and is halved during each 25-year period, so



From this pattern, it appears that the mass remaining after t years is



This is an exponential function with base

1. The mass that remains after 40 years is
2. We use a graphing calculator or computer to graph the function in Figure 12. We also graph the line m = 5 and use the cursor to estimate that when. So the mass of the sample will be reduced to 5 mg after about 57 years.

### The Number e

Of all possible bases for an exponential function, there is one that is most convenient for the purposes of calculus. The choice of a base b is influenced by the way the graph of crosses the y-axis. Figures 13 and 14 show the tangent lines to the graphs of andat the point (0, 1). (Tangent lines will be defined precisely in Section 2.7. For present purposes, you can think of the tangent line to an exponential graph at a point as the line that touches the graph only at that point.) If we measure the slopes of these tangent lines at (0, 1), we find that for  and for.

FIGURE 13:

FIGURE 14:

It turns out, as we will see in Chapter 3, that some of the formulas of calculus will be greatly simplified if we choose the base b so that the slope of the tangent line to 

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at (0, 1) is exactly 1. (See Figure 15.) In fact, there is such a number and it is denoted by the letter . (This notation was chosen by the Swiss mathematician Leonhard Euler in 1727, probably because it is the first letter of the word exponential.) In view of Figures 13 and 14, it comes as no surprise that the number e lies between 2 and 3 and the graph of lies between the graphs ofand. (See Figure 16.) In Chapter 3 we will see that the value of e, correct to five decimal places, is



We call the function the natural exponential function.

FIGURE 15:

The natural exponential function crosses the y-axis with a slope of 1.

TEC Module 1.4 enables you to graph exponential functions with various bases and their tangent lines in order to estimate more closely the value of b for which the tangent has slope 1.

FIGURE 16

## EXAMPLE 4

Graph the function and state the domain and range.

SOLUTION We start with the graph of from Figures 15 and 17(a) and reflect about the y-axis to get the graph of in Figure 17(b). (Notice that the graph crosses the y-axis with a slope of -1). Then we compress the graph vertically by a factor of 2 to obtain the graph of in Figure 17(c). Finally, we shift the graph downward one unit to get the desired graph in Figure 17(d). The domain is and the range is.

FIGURE 17:

1. 
2. 
3. 
4. 

How far to the right do you think we would have to go for the height of the graph of  to exceed a million? The next example demonstrates the rapid growth of this function by providing an answer that might surprise you.

## EXAMPLE 5

Use a graphing device to find the values of x for which.

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SOLUTION In Figure 18 we graph both the function and the horizontal line y = 1,000,000. We see that these curves intersect when . Thus when x >13.8. It is perhaps surprising that the values of the exponential function have already surpassed a million when x is only 14.

FIGURE 18: 

## 1.4 Exercises

1–4 Use the Law of Exponents to rewrite and simplify the expression.

* 1. 
  2. 
  3. 
  4. 
  5. 
  6. 
  7. 
  8. 
  9. Write an equation that defines the exponential function with base b > 0.
  10. What is the domain of this function?
  11. If , what is the range of this function?
  12. Sketch the general shape of the graph of the exponential function for each of the following cases
      1. 
      2. 
      3. 
  13. How is the number e defined?
  14. What is an approximate value for e?
  15. What is the natural exponential function?

7-10 Graph the given functions on a common screen. How are these graphs related?

1. 
2. 
3. 
4. 

11-16 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graphs given in Figures 3 and 13 and, if necessary, the transformations of Section 1.3.

1. 
2. 
3. 
4. 
5. 
6. 
7. Starting with the graph of, write the equation of the graph that results form
   1. Shifting 2 units downward.
   2. Shifting 2 units to the right.
   3. Reflecting about the x-axis.
   4. Reflecting about the y-axis
   5. Reflecting about the x-axis and then about the y-axis.
8. Starting with the graph of , find the equation of the graph that results from
   1. Reflecting about the line y = 4.
   2. Reflecting about the line x = 2.

19-20 Find the domain of each function.

* 1. 
  2. 
  3. 
  4. 

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21-22 Find the exponential functionwhose graph is given.

1. See graph tactile
2. See graph tactile
3. If, show that
4. Suppose you are offered a job that lasts one month. Which of the following methods of payment do you prefer?
   1. One million dollars at the end of the month
   2. One cent on the first day of the month, two cents on the second day, four cents on the third day, and in general, cents on the nth day.
5. Suppose the graphs ofandare drawn on a coordinate grid where the unit of measurement is 1 inch. Show that, at a distance 2 ft to the right of the origin, the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.
6. Compare the functions andby graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place. Which function grows more rapidly when x is large?
7. Compare the functionandby graphing both f and g in several viewing rectangles. When does the graph of g finally surpass the graph of f?
8. Use a graph to estimate the values of x such that
9. A researcher is trying to determine the doubling time for a population of the bacterium Giardia lamblia. He starts a culture in a nutrient solution and estimates the bacteria count every four hours. His data are shown in the table.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Time (hours) | 0 | 4 | 8 | 12 | 16 | 20 | 24 |
| Bacteria Count (CFU/ mL) | 37 | 47 | 63 | 78 | 105 | 130 | 173 |

* 1. Make a scatter plot of the data.
  2. Use a graphing calculator to find an exponential curve that models the bacteria population t hours later.
  3. Graph the model from part (a) together with the scatter plot in part (a). Use the TRACE feature to determine how long it takes for the bacteria count to double.

1. A bacteria culture starts with 500 bacteria and doubles in size every half hour.
   1. How many bacteria are there after 3 hours?
   2. How many bacteria are there after t hours?
   3. How many bacteria are there after 40 minutes?
   4. Graph the population function and estimate the time for the population to reach 100,000.
2. The half-life of bismuth-210, , is 5 days.
   1. If a sample has a mass of 200 mg, find the amount remaining after 15 days.
   2. Find the amount remaining after t days.
   3. Estimate the amount remaining after 3 weeks.
   4. Use a graph to estimate the time required for the mass to be reduced to 1 mg.
3. An isotope of sodium, , has a half-life of 15 hours. A sample of this isotope has mass 2 g.
   1. Find the amount remaining after 60 hours.
   2. Find the amount remaining after t hours.
   3. Estimate the amount remaining after 4 days.
   4. Use a graph to estimate the time required for the mass to be reduced to 0.01 g.
4. Use the graph of V in Figure 11 to estimate the half-life of the viral load of patient 303 during the first month of treatment.
5. After alcohol is fully absorbed into the body, it is metabolized with a half-life of about 1.5 hours. Suppose you have had three alcoholic drinks and an hour later, at midnight, your blood alcohol concentration (BAC) is 0.6 mg/mL.
   1. Find an exponential decay model for your BAC t hours after midnight.
   2. Graph your BAC and use the graph to determine when your BAC is 0.08 mg/mL.
6. Use a graphing calculator with exponential regression capability to model the population of the world with the date from 1950 to 2010 in Table 1 on page 49. Use the model to estimate the population in 1993 and to predict the population in the year 2020.

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1. The table gives the population of the United States, in millions, for the years 1900-2010. Use a graphing calculator with exponential regression capability to model the US population since 1900. Use the model to estimate the population in 1925 and to predict the population in the year 2020.

|  |  |
| --- | --- |
| Year | Population |
| 1900 | 76 |
| 1910 | 92 |
| 1920 | 106 |
| 1930 | 123 |
| 1940 | 131 |
| 1950 | 150 |
| 1960 | 179 |
| 1970 | 203 |
| 1980 | 227 |
| 1990 | 250 |
| 2000 | 281 |
| 2010 | 310 |

1. If you graph the functionyou’ll see the f appears to be an off function. Prove it.
2. Graph several members of the family of functionswhere a > 0. How does the graph change when b changes? How does it change when a changes?