4.3 Cosine and Sine

**Learning Objectives**

By the end of this section you should be able to

- evaluate the cosine and sine of any multiple of 30° or 45° ($\frac{\pi}{6}$ radians or $\frac{\pi}{4}$ radians);
- determine whether the cosine (or sine) of an angle is positive or negative from the location of the corresponding radius;
- sketch the radius corresponding to $\theta$ if given either $\cos \theta$ or $\sin \theta$ and the sign of the other quantity;
- find $\cos \theta$ and $\sin \theta$ if given one of these quantities and the quadrant of the corresponding radius.

**Definition of Cosine and Sine**

The table here shows the endpoint of the radius of the unit circle corresponding to some special angles. This table comes from tables in Sections 4.1 and 4.2.

<table>
<thead>
<tr>
<th>$\theta$ (radians)</th>
<th>$\theta$ (degrees)</th>
<th>endpoint of radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0°</td>
<td>(1, 0)</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>30°</td>
<td>($\frac{\sqrt{3}}{2}$, $\frac{1}{2}$)</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>45°</td>
<td>($\frac{\sqrt{2}}{2}$, $\frac{\sqrt{2}}{2}$)</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>60°</td>
<td>($\frac{1}{2}$, $\frac{\sqrt{3}}{2}$)</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>90°</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>$\pi$</td>
<td>180°</td>
<td>(−1, 0)</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>360°</td>
<td>(1, 0)</td>
</tr>
</tbody>
</table>

Coordinates of the endpoint of the radius of the unit circle corresponding to special angles.

If no units are given for an angle, then assume the units are radians.

The figure below shows a radius of the unit circle corresponding to $\theta$ (here $\theta$ might be measured in either radians or degrees). The endpoint of this radius is used to define the cosine and sine, as follows.

**Cosine**

The **cosine** of $\theta$, denoted $\cos \theta$, is the first coordinate of the endpoint of the radius of the unit circle corresponding to $\theta$.

**Sine**

The **sine** of $\theta$, denoted $\sin \theta$, is the second coordinate of the endpoint of the radius of the unit circle corresponding to $\theta$.

The two definitions above can be combined into a single statement, as follows.

**Cosine and sine**

The endpoint of the radius of the unit circle corresponding to $\theta$ has coordinates $(\cos \theta, \sin \theta)$. 

This figure defines cosine and sine. If you understand this figure well, then you can figure out a big chunk of trigonometry.
**Example 1**

Evaluate \( \cos \frac{\pi}{2} \) and \( \sin \frac{\pi}{2} \).

**solution** The radius corresponding to \( \frac{\pi}{2} \) radians has endpoint \((0,1)\). Thus

\[
\cos \frac{\pi}{2} = 0 \quad \text{and} \quad \sin \frac{\pi}{2} = 1.
\]

Using degrees instead of radians, we could write \( \cos 90^\circ = 0 \) and \( \sin 90^\circ = 1 \).

The table below gives the cosine and sine of some special angles. This table is obtained by breaking the last column of the previous table into two columns, with the first coordinate labeled as cosine and the second coordinate labeled as sine.

<table>
<thead>
<tr>
<th>( \theta ) (radians)</th>
<th>( \theta ) (degrees)</th>
<th>( \cos \theta )</th>
<th>( \sin \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0^\circ</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{6} )</td>
<td>30^\circ</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>45^\circ</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
</tr>
<tr>
<td>( \frac{\pi}{3} )</td>
<td>60^\circ</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>90^\circ</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \pi )</td>
<td>180^\circ</td>
<td>( -1 )</td>
<td>0</td>
</tr>
<tr>
<td>( 2\pi )</td>
<td>360^\circ</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The next example extends the table above to another special angle.

**Example 2**

Evaluate \( \cos \left( -\frac{\pi}{2} \right) \) and \( \sin \left( -\frac{\pi}{2} \right) \).

**solution** The radius corresponding to \( -\frac{\pi}{2} \) radians has endpoint \((0,-1)\) as shown here. Thus

\[
\cos \left( -\frac{\pi}{2} \right) = 0 \quad \text{and} \quad \sin \left( -\frac{\pi}{2} \right) = -1.
\]

Using degrees instead of radians, we have \( \cos(-90^\circ) = 0 \) and \( \sin(-90^\circ) = -1 \).

In addition to adding a row for \( -\frac{\pi}{2} \) radians (which equals \(-90^\circ\)), we could add many more entries to the table for the cosine and sine of special angles. Possibilities would include \( \frac{2\pi}{3} \) radians (which equals 120^\circ), \( \frac{5\pi}{6} \) radians (which equals 150^\circ), the negatives of all the angles already in the table, and so on. This would quickly become far too much information to memorize. Instead of memorizing, concentrate on understanding the definitions of cosine and sine.

Similarly, do not become dependent on a calculator for evaluating the cosine and sine of special angles. If you need numeric values for \( \cos 2 \) or \( \sin 17^\circ \), then use a calculator. But if you get in the habit of using a calculator for evaluating expressions such as \( \cos 0 \) or \( \sin(-180^\circ) \), then cosine and sine will become simply buttons on your calculator and you will not be able to use these functions meaningfully.

Note that \( \cos \) and \( \sin \) are functions; thus \( \cos(\theta) \) and \( \sin(\theta) \) might be a better notation than \( \cos \theta \) and \( \sin \theta \). In an expression such as

\[
\frac{\cos 10}{\cos 5},
\]

we cannot cancel \( \cos \) in the numerator and denominator, just as we cannot cancel a function \( f \) in the numerator and denominator of \( \frac{f(10)}{f(5)} \). Similarly, the expression above is not equal to \( \cos 2 \), just as \( \frac{f(10)}{f(5)} \) is usually not equal to \( f(2) \).
The Signs of Cosine and Sine

The coordinate axes divide the coordinate plane into four regions, often called quadrants. The quadrant in which a radius lies determines whether the cosine and sine of the corresponding angle are positive or negative. The figure below shows the sign of the cosine and the sign of the sine in each of the four quadrants. Thus, for example, an angle corresponding to a radius lying in the region marked “cos θ < 0, sin θ > 0” (the upper-left quadrant) will have a cosine that is negative and a sine that is positive.

Recall that the cosine of an angle is the first coordinate of the endpoint of the corresponding radius. Thus the cosine is positive in the two quadrants where the first coordinate is positive, as shown in the figure above. Also, the cosine is negative in the two quadrants where the first coordinate is negative.

Similarly, the sine of an angle is the second coordinate of the endpoint of the corresponding radius. Thus the sine is positive in the two quadrants where the second coordinate is positive, as shown in the figure above. Also, the sine is negative in the two quadrants where the second coordinate is negative.

The next example should help you understand how the quadrant determines the sign of the cosine and sine.

Example 3

(a) Evaluate cos \(\frac{\pi}{4}\) and sin \(\frac{\pi}{4}\).

(b) Evaluate cos \(\frac{3\pi}{4}\) and sin \(\frac{3\pi}{4}\).

(c) Evaluate cos(−\(\frac{\pi}{4}\)) and sin(−\(\frac{\pi}{4}\)).

(d) Evaluate cos(−\(\frac{3\pi}{4}\)) and sin(−\(\frac{3\pi}{4}\)).

solution The four angles \(\frac{\pi}{4}\), \(\frac{3\pi}{4}\), −\(\frac{\pi}{4}\), and −\(\frac{3\pi}{4}\) radians (or, equivalently, 45°, 135°, −45°, and −135°) are shown in the next figure. Each coordinate of the radius corresponding to each of these angles is either \(\sqrt{2}\) or −\(\sqrt{2}\); the only issue to worry about in computing the cosine and sine of these angles is the sign.
(a) Both coordinates of the endpoint of the radius corresponding to \( \frac{\pi}{4} \) radians are positive. Thus \( \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \) and \( \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \).

(b) The first coordinate of the endpoint of the radius corresponding to \( \frac{3\pi}{4} \) radians is negative; the second coordinate is positive. Thus \( \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2} \) and \( \sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2} \).

(c) The first coordinate of the endpoint of the radius corresponding to \( -\frac{\pi}{4} \) radians is positive and the second coordinate is negative. Thus \( \cos \left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \) and \( \sin \left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \).

(d) Both coordinates of the endpoint of the radius corresponding to \( -\frac{3\pi}{4} \) radians are negative. Thus \( \cos \left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} \) and \( \sin \left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} \).

The next example shows how to use information about the signs of the cosine and sine to locate the corresponding radius.

**Example 4**

Sketch the radius of the unit circle corresponding to an angle \( \theta \) such that \( \cos \theta = 0.4 \) and \( \sin \theta < 0 \).

**Solution** Because \( \cos \theta \) is positive and \( \sin \theta \) is negative, the radius corresponding to \( \theta \) lies in the lower-right quadrant. To find the endpoint of this radius, which has first coordinate 0.4, start with the point 0.4 on the horizontal axis and then move vertically down to reach a point on the unit circle. Then draw the radius from the origin to that point, as shown here.

The Key Equation Connecting Cosine and Sine

By definition of cosine and sine, the point \((\cos \theta, \sin \theta)\) is on the unit circle, which is the set of points in the coordinate plane such that the sum of the squares of the coordinates equals 1. In the \(xy\)-plane, the unit circle is described by the equation

\[
x^2 + y^2 = 1.
\]

Thus the following crucial equation holds.

**Relationship between cosine and sine**

\[
(cos \theta)^2 + (sin \theta)^2 = 1
\]

for every angle \( \theta \).
Given either $\cos \theta$ or $\sin \theta$, the last equation can be used to solve for the other quantity, provided that we have enough additional information to determine the sign. The following example illustrates this procedure.

**Example 5**

Suppose $\theta$ is an angle such that $\sin \theta = 0.6$, and suppose also that $\frac{\pi}{2} < \theta < \pi$. Evaluate $\cos \theta$.

**solution**  The equation above implies that $(\cos \theta)^2 + (0.6)^2 = 1$. Because $(0.6)^2 = 0.36$, this implies that

$$(\cos \theta)^2 = 0.64.$$  

Thus $\cos \theta = 0.8$ or $\cos \theta = -0.8$. The additional information that $\frac{\pi}{2} < \theta < \pi$ implies that $\cos \theta$ is negative, as can been seen in the figure. Thus

$$\cos \theta = -0.8.$$  

**The Graphs of Cosine and Sine**

Before graphing the cosine and sine functions, we should think carefully about the domain and range of these functions. Recall that for each real number $\theta$, there is a radius of the unit circle corresponding to $\theta$.

Recall also that the coordinates of the endpoints of the radius corresponding to the angle $\theta$ are labeled $(\cos \theta, \sin \theta)$, thus defining the cosine and sine functions. These functions are defined for every real number $\theta$. Thus the domain of both cosine and sine is the set of real numbers.

As we have already noted, a consequence of $(\cos \theta, \sin \theta)$ lying on the unit circle is the equation

$$(\cos \theta)^2 + (\sin \theta)^2 = 1.$$  

Because $(\cos \theta)^2$ and $(\sin \theta)^2$ are both nonnegative, the equation above implies that

$$(\cos \theta)^2 \leq 1 \quad \text{and} \quad (\sin \theta)^2 \leq 1.$$  

Thus $\cos \theta$ and $\sin \theta$ must both be between $-1$ and $1$.

These inequalities could also be written in the following form:

$$|\cos \theta| \leq 1 \quad \text{and} \quad |\sin \theta| \leq 1.$$  

The first coordinates of the points of the unit circle are precisely the values of the cosine function. Every number in the interval $[-1,1]$ is the first coordinate of some point on the unit circle. Thus we can conclude that the range of the cosine function is the interval $[-1,1]$. A similar conclusion holds for the sine function (use second coordinates instead of first coordinates).

In summary, we have the following results.

**Domain and range of cosine and sine**

- The domain of both cosine and sine is the set of real numbers.
- The range of both cosine and sine is the interval $[-1,1]$.
Because the domain of the cosine and the sine is the set of real numbers, we cannot show the graph of these functions on their entire domain. To understand what the graphs of these functions look like, we start by looking at the graph of cosine on the interval $[-6\pi, 6\pi]$.

Let’s begin examining the graph above by noting that the point $(0, 1)$ is on the graph, as expected from the equation $\cos 0 = 1$. Note that the horizontal axis has been called the $\theta$-axis.

Moving to the right along the $\theta$-axis from the origin, we see that the graph crosses the $\theta$-axis at the point $(\frac{\pi}{2}, 0)$, as expected from the equation $\cos \frac{\pi}{2} = 0$. Continuing further to the right, we see that the graph hits its lowest value when $\theta = \pi$, as expected from the equation $\cos \pi = -1$. The graph then crosses the $\theta$-axis again at the point $(\frac{3\pi}{2}, 0)$, as expected from the equation $\cos \frac{3\pi}{2} = 0$. Then the graph hits its highest value again when $\theta = 2\pi$, as expected from the equation $\cos (2\pi) = 1$.

The most striking feature of the graph above is its periodic nature—the graph repeats itself. To understand why the graph of cosine exhibits this periodic behavior, consider a radius of the unit circle starting along the positive horizontal axis and moving counterclockwise. As the radius moves, the first coordinate of its endpoint gives the value of the cosine of the corresponding angle. After the radius moves through an angle of $2\pi$ radians, it returns to its original position. Then it begins the cycle again, returning to its original position after moving through a total angle of $4\pi$, and so on. Thus we see the periodic behavior of the graph of cosine.

In Section 4.6 we will examine the properties of cosine and its graph more deeply. For now, let’s turn to the graph of sine. Here is the graph of sine on the interval $[-6\pi, 6\pi]$.

This graph goes through the origin, as expected because $\sin 0 = 0$. Moving to the right along the $\theta$-axis from the origin, we see that the graph hits its highest value when $\theta = \frac{\pi}{2}$, as expected because $\sin \frac{\pi}{2} = 1$. Continuing further to the right, we see that the graph crosses the $\theta$-axis at the point $(\pi, 0)$, as expected because $\sin \pi = 0$. The graph then hits its lowest value when $\theta = \frac{3\pi}{2}$, as expected because $\sin \frac{3\pi}{2} = -1$. Then the graph crosses the $\theta$-axis again at $(2\pi, 0)$, as expected because $\sin (2\pi) = 0$.

Surely you have noticed that the graph of sine looks much like the graph of cosine. It appears that shifting one graph somewhat to the left or right produces the other graph. We will see that this is indeed the case when we delve more deeply into properties of cosine and sine in Section 4.6.
Suppose you have borrowed two calculators from friends, but you do not know whether they are set to work in radians or degrees. Thus you ask each calculator to evaluate $\cos 3.14$. One calculator gives an answer of 0.999999; the other calculator gives an answer of 0.998499. Without further use of a calculator, how would you decide which calculator is using radians and which calculator is using degrees? Explain your answer.

Suppose you have borrowed two calculators from friends, but you do not know whether they are set to work in radians or degrees. Thus you ask each calculator to evaluate $\sin 1$. One calculator gives an answer of 0.841471; the other calculator gives an answer of 0.017452. Without further use of a calculator, how would you decide which calculator is using radians and which calculator is using degrees? Explain your answer.

Give exact values for the quantities in Exercises 1–10. Do not use a calculator for any of these exercises—otherwise you will likely get decimal approximations for some solutions rather than exact answers. More importantly, good understanding will come from working these exercises by hand.

Exercises

1. Find the four smallest positive numbers $\theta$ such that $\sin \theta = \frac{1}{2}$.
2. Find the four smallest positive numbers $\theta$ such that $\cos \theta = \frac{1}{2}$.
3. Suppose $0 < \theta < \frac{\pi}{2}$ and $\cos \theta = \frac{3}{4}$. Evaluate $\sin \theta$.
4. Suppose $0 < \theta < \frac{\pi}{2}$ and $\sin \theta = \frac{3}{4}$. Evaluate $\cos \theta$.
5. Suppose $\frac{\pi}{2} < \theta < \pi$ and $\sin \theta = \frac{3}{4}$. Evaluate $\cos \theta$.
6. Suppose $\frac{\pi}{2} < \theta < \pi$ and $\cos \theta = \frac{3}{4}$. Evaluate $\sin \theta$.
7. Suppose $\frac{\pi}{2} < \theta < 0$ and $\cos \theta = 0.1$. Evaluate $\sin \theta$.
8. Suppose $\frac{\pi}{2} < \theta < 0$ and $\sin \theta = 0.3$. Evaluate $\cos \theta$.
9. Find the smallest number $x$ such that $\sin(x^2) = 0$.
10. Find the smallest number $x$ such that $\cos(x^2 + 1) = 0$.

Problems

35. Sketch a radius of the unit circle corresponding to an angle $\theta$ such that $\cos \theta = \frac{6}{7}$.
36. Sketch another radius, different from the one in part (a), also illustrating $\cos \theta = \frac{6}{7}$.
37. Find angles $u$ and $v$ such that $\cos u = \cos v$ but $\sin u \neq \sin v$.
38. Find angles $u$ and $v$ such that $\sin u = \sin v$ but $\cos u \neq \cos v$.
39. Show that $\ln(\cos \theta)$ is the average of $\ln(1 - \sin \theta)$ and $\ln(1 + \sin \theta)$ for every $\theta$ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
42 A good scientific calculator will show that
\[ \cos 710 \approx 0.999999998, \]
where of course the left side means the cosine of 710 radians. Thus \( \cos 710 \) is remarkably close to 1. Use the approximation \( \pi \approx \frac{355}{113} \) (which has an error of less than \( 3 \times 10^{-7} \)) to explain why \( \cos 710 \approx 1 \).

43 Suppose \( m \) is a real number. Let \( \theta \) be the acute angle between the positive horizontal axis and the line with slope \( m \) through the origin. Evaluate \( \cos \theta \) and \( \sin \theta \).

44 Explain why there does not exist a real number \( x \) such that \( 2 \sin x = \frac{3}{7} \).

45 Explain why \( \pi \cos x < 4 \) for every real number \( x \).

46 Explain why \( \frac{1}{3} < e^{\sin x} \) for every number real number \( x \).

47 Explain why the equation \( (\sin x)^2 - 4 \sin x + 4 = 0 \) has no solutions.

48 Explain why the equation \( (\cos x)^{99} + 4 \cos x - 6 = 0 \) has no solutions.

49 Explain why there does not exist a number \( \theta \) such that \( \log \cos \theta = 0.1 \).

Worked-Out Solutions to Odd-Numbered Exercises

Give exact values for the quantities in Exercises 1–10. Do not use a calculator for any of these exercises—otherwise you will likely get decimal approximations for some solutions rather than exact answers. More importantly, good understanding will come from working these exercises by hand.

1 (a) \( \cos(3\pi) \)  \hspace{1cm} (b) \( \sin(3\pi) \)

**solution** Because \( 3\pi = 2\pi + \pi \), an angle of \( 3\pi \) radians (as measured counterclockwise from the positive horizontal axis) consists of a complete revolution around the circle (\( 2\pi \) radians) followed by another \( \pi \) radians (180°), as shown below. The endpoint of the corresponding radius is \((-1, 0)\). Thus \( \cos(3\pi) = -1 \) and \( \sin(3\pi) = 0 \).

3 (a) \( \cos \frac{11\pi}{4} \)  \hspace{1cm} (b) \( \sin \frac{11\pi}{4} \)

**solution** Because \( \frac{11\pi}{4} = 2\pi + \frac{\pi}{4} \), an angle of \( \frac{11\pi}{4} \) radians (as measured counterclockwise from the positive horizontal axis) consists of a complete revolution around the circle (\( 2\pi \) radians) followed by another \( \frac{\pi}{4} \) radians (45°), followed by another \( \frac{\pi}{4} \) radians (45°), as shown below. Hence the endpoint of the corresponding radius is \((-\sqrt{2}, \sqrt{2})\). Thus \( \cos \frac{11\pi}{4} = -\frac{\sqrt{2}}{2} \) and \( \sin \frac{11\pi}{4} = \frac{\sqrt{2}}{2} \).

5 (a) \( \cos \frac{2\pi}{3} \)  \hspace{1cm} (b) \( \sin \frac{2\pi}{3} \)

**solution** Because \( \frac{2\pi}{3} = \frac{\pi}{2} + \frac{\pi}{6} \), an angle of \( \frac{2\pi}{3} \) radians (as measured counterclockwise from the positive horizontal axis) consists of \( \frac{\pi}{2} \) radians (90°) followed by another \( \frac{\pi}{6} \) radians (30°), as shown below. The endpoint of the corresponding radius is \((-\frac{1}{2}, \frac{\sqrt{3}}{2})\). Thus \( \cos \frac{2\pi}{3} = -\frac{1}{2} \) and \( \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \).

7 (a) \( \cos 210^\circ \)  \hspace{1cm} (b) \( \sin 210^\circ \)

**solution** Because \( 210 = 180 + 30 \), an angle of \( 210^\circ \) (as measured counterclockwise from the positive horizontal axis) consists of \( 180^\circ \) followed by another \( 30^\circ \), as shown below. The endpoint of the corresponding radius is \((-\frac{\sqrt{3}}{2}, -\frac{1}{2})\). Thus \( \cos 210^\circ = -\frac{\sqrt{3}}{2} \) and \( \sin 210^\circ = -\frac{1}{2} \).
Find the smallest number \( \theta \) larger than \( 4\pi \) such that \( \cos \theta = 0 \).

**solution** Note that

\[
0 = \cos \frac{\pi}{2} = \cos \frac{3\pi}{2} = \cos \frac{5\pi}{2} = \ldots
\]

and that the only numbers whose cosine equals 0 are of the form \( \frac{(2n+1)\pi}{2} \), where \( n \) is an integer. The smallest number of this form larger than \( 4\pi \) is \( \frac{9\pi}{2} \). Thus \( \frac{9\pi}{2} \) is the smallest number larger than \( 4\pi \) whose cosine equals 0.

Find the four smallest positive numbers \( \theta \) such that \( \sin \theta = 0 \).

**solution** Think of a radius of the unit circle whose endpoint is \( (1,0) \). If this radius moves counterclockwise, forming an angle of \( \theta \) radians with the positive horizontal axis, the second coordinate of its endpoint first becomes 0 when \( \theta \) equals \( \frac{\pi}{2} \) (which equals 90\(^\circ\)), then again when \( \theta \) equals \( \frac{3\pi}{2} \) (which equals 270\(^\circ\)), then again when \( \theta \) equals \( \frac{5\pi}{2} \) (which equals 360\(^\circ\) + 90\(^\circ\), or 450\(^\circ\)), then again when \( \theta \) equals \( \frac{7\pi}{2} \) (which equals 360\(^\circ\) + 270\(^\circ\), or 630\(^\circ\)), and so on. Thus the four smallest positive numbers \( \theta \) such that \( \sin \theta = 1 \) are \( \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2} \), and \( \frac{13\pi}{2} \).

Find the four smallest positive numbers \( \theta \) such that \( \cos \theta = -1 \).

**solution** Think of a radius of the unit circle whose endpoint is \( (1,0) \). If this radius moves counterclockwise, forming an angle of \( \theta \) radians with the positive horizontal axis, the first coordinate of its endpoint first becomes 0 when \( \theta \) equals \( \pi \) (which equals 180\(^\circ\)), then again when \( \theta \) equals \( 3\pi \) (which equals 360\(^\circ\) + 180\(^\circ\), or 540\(^\circ\)), then again when \( \theta \) equals \( 5\pi \) (which equals \( 2 \times 360\(^\circ\) + 180\(^\circ\), or 900\(^\circ\)), then again when \( \theta \) equals \( 7\pi \) (which equals \( 3 \times 360\(^\circ\) + 180\(^\circ\), or 1260\(^\circ\)), and so on. Thus the four smallest positive numbers \( \theta \) such that \( \cos \theta = -1 \) are \( \pi, 3\pi, 5\pi, \) and \( 7\pi \).

Suppose \( 0 < \theta < \frac{\pi}{2} \) and \( \cos \theta = \frac{1}{2} \). Evaluate \( \sin \theta \).

**solution** We know that

\[
(cos \theta)^2 + (sin \theta)^2 = 1.
\]

Thus

\[
(sin \theta)^2 = 1 - (cos \theta)^2 = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}.
\]

Because \( 0 < \theta < \frac{\pi}{2} \), we know that \( \sin \theta > 0 \). Thus taking square roots of both sides of the equation above gives

\[
\sin \theta = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.
\]
Suppose \( \frac{\pi}{2} < \theta < \pi \), we know that \( \cos \theta < 0 \). Thus taking square roots of both sides of the equation above gives
\[
\cos \theta = -\sqrt{77} = 8.775.
\]

Because \( \pi < \theta < 2\pi \), we know that \( \sin \theta < 0 \). Thus taking square roots of both sides of the equation above gives
\[
\sin \theta = -\sqrt{0.99} \approx -0.995.
\]

29 Find the smallest positive number \( x \) such that
\[
\sin(x^2 + x + 4) = 0.
\]

**solution** Note that \( e^x \) is an increasing function. Because \( e^x \) is positive for every real number \( x \), and because \( \pi \) is the smallest positive number whose sine equals 0, we want to choose \( x \) so that \( e^x = \pi \). Thus \( x = \ln \pi \).

31 Let \( \theta \) be the acute angle between the positive horizontal axis and the line with slope 3 through the origin. Evaluate \( \cos \theta \) and \( \sin \theta \).

**solution** From the solution to Exercise 5 in Section 4.1, we see that the endpoint of the relevant radius on the unit circle has coordinates \( \left( \frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10} \right) \). Thus
\[
\cos \theta = \frac{\sqrt{10}}{10} \quad \text{and} \quad \sin \theta = \frac{3\sqrt{10}}{10}.
\]

33 Suppose \( 5\cos \theta^2 - 7 \cos \theta - 6 = 0 \). Evaluate \( |\sin \theta| \).

**solution** Let \( y = \cos \theta \). The equation above can be rewritten as
\[
5y^2 - 7y - 6 = 0.
\]

Solving for \( y \) (using either the quadratic formula or by factoring the left side of the equation above), we have
\[
y = -\frac{3}{5} \quad \text{or} \quad y = 2.
\]

However, \( y = \cos \theta \), and there does not exist a real number \( \theta \) such that \( \cos \theta = 2 \). Hence \( y = -\frac{3}{5} \), and hence \( \cos \theta = -\frac{3}{5} \). Now
\[
(\sin \theta)^2 = 1 - (\cos \theta)^2
= 1 - \left( -\frac{3}{5} \right)^2
= 1 - \frac{9}{25}
= \frac{16}{25}.
\]

Thus \( \sin \theta = \pm \sqrt{\frac{16}{25}} = \pm \frac{4}{5} \). Thus \( |\sin \theta| = \frac{4}{5} \).