4.3 Cosine and Sine

Learning Objectives

By the end of this section you should be able to

- evaluate the cosine and sine of any multiple of 30° or 45° ($\frac{\pi}{6}$ radians or $\frac{\pi}{4}$ radians);
- determine whether the cosine (or sine) of an angle is positive or negative from the location of the corresponding radius;
- sketch the radius corresponding to θ if given either $\cos\theta$ or $\sin\theta$ and the sign of the other quantity;
- find $\cos\theta$ and $\sin\theta$ if given one of these quantities and the quadrant of the corresponding radius.

Definition of Cosine and Sine

The table here shows the endpoint of the radius of the unit circle corresponding to some special angles. This table comes from tables in Sections 4.1 and 4.2.

We might consider extending the table above to other angles. For example, suppose we want to know the endpoint of the radius corresponding to $\frac{\pi}{18}$ radians. Unfortunately, the coordinates of the endpoint of that radius do not have a nice form—neither coordinate is a rational number or even the square root of a rational number. The cosine and sine functions, which we are about to introduce, were invented to help us extend this table to all angles.

θ radians	θ degrees	endpoint of radius
0	0°	$(1, 0)$
$\frac{\pi}{6}$	30°	$(\frac{\sqrt{3}}{2}, \frac{1}{2})$
$\frac{\pi}{4}$	45°	$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
$\frac{\pi}{3}$	60°	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
$\frac{\pi}{2}$	90°	$(0, 1)$
π	180°	$(-1, 0)$
2π	360°	$(1, 0)$

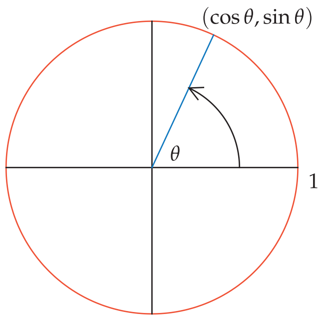
Coordinates of the endpoint of the radius of the unit circle corresponding to special angles.

Before introducing the cosine and sine functions, we state a common assumption about notation in trigonometry.

Angles without units

If no units are given for an angle, then assume the units are radians.

The figure below shows a radius of the unit circle corresponding to θ (here θ might be measured in either radians or degrees). The endpoint of this radius is used to define the cosine and sine, as follows.



This figure defines cosine and sine. If you understand this figure well, then you can figure out a big chunk of trigonometry.

Cosine

The **cosine** of θ, denoted $\cos\theta$, is the first coordinate of the endpoint of the radius of the unit circle corresponding to θ.

Sine

The **sine** of θ, denoted $\sin\theta$, is the second coordinate of the endpoint of the radius of the unit circle corresponding to θ.

The two definitions above can be combined into a single statement, as follows.

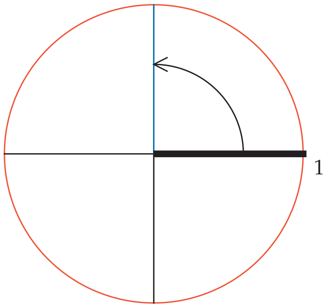
Cosine and sine

The endpoint of the radius of the unit circle corresponding to θ has coordinates $(\cos\theta, \sin\theta)$.

Example 1

Evaluate $\cos\frac{\pi}{2}$ and $\sin\frac{\pi}{2}$.

Here units are not specified for the angle $\frac{\pi}{2}$. Thus we assume we are dealing with $\frac{\pi}{2}$ radians.



The radius corresponding to $\frac{\pi}{2}$ radians has endpoint $(0, 1)$.

solution The radius corresponding to $\frac{\pi}{2}$ radians has endpoint $(0, 1)$. Thus

$$\cos\frac{\pi}{2} = 0 \quad \text{and} \quad \sin\frac{\pi}{2} = 1.$$

Using degrees instead of radians, we could write $\cos 90° = 0$ and $\sin 90° = 1$.

The table below gives the cosine and sine of some special angles. This table is obtained by breaking the last column of the previous table into two columns, with the first coordinate labeled as cosine and the second coordinate labeled as sine.

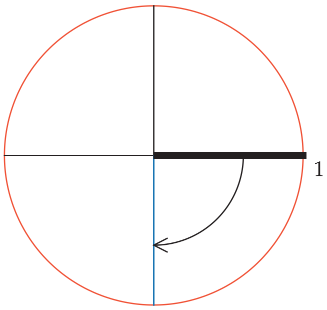
Compare this table to the first table in this section and make sure you understand what is going on here.

θ (*radians*)	θ (*degrees*)	$\cos\theta$	$\sin\theta$
0	$0°$	1	0
$\frac{\pi}{6}$	$30°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$60°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	$90°$	0	1
π	$180°$	-1	0
2π	$360°$	1	0

The next example extends the table above to another special angle.

Example 2

Evaluate $\cos\left(-\frac{\pi}{2}\right)$ and $\sin\left(-\frac{\pi}{2}\right)$.



The radius corresponding to $-\frac{\pi}{2}$ radians has endpoint $(0, -1)$.

solution The radius corresponding to $-\frac{\pi}{2}$ radians has endpoint $(0, -1)$ as shown here. Thus

$$\cos\left(-\frac{\pi}{2}\right) = 0 \quad \text{and} \quad \sin\left(-\frac{\pi}{2}\right) = -1.$$

Using degrees instead of radians, we have $\cos(-90°) = 0$ and $\sin(-90°) = -1$.

In addition to adding a row for $-\frac{\pi}{2}$ radians (which equals $-90°$), we could add many more entries to the table for the cosine and sine of special angles. Possibilities would include $\frac{2\pi}{3}$ radians (which equals $120°$), $\frac{5\pi}{6}$ radians (which equals $150°$), the negatives of all the angles already in the table, and so on. This would quickly become far too much information to memorize. Instead of memorizing, concentrate on understanding the definitions of cosine and sine.

Most calculators can work in radians or degrees. When you use a calculator to compute values of cosine or sine, be sure your calculator is set to work in the appropriate units.

Similarly, do not become dependent on a calculator for evaluating the cosine and sine of special angles. If you need numeric values for $\cos 2$ or $\sin 17°$, then use a calculator. But if you get in the habit of using a calculator for evaluating expressions such as $\cos 0$ or $\sin(-180°)$, then cosine and sine will become simply buttons on your calculator and you will not be able to use these functions meaningfully.

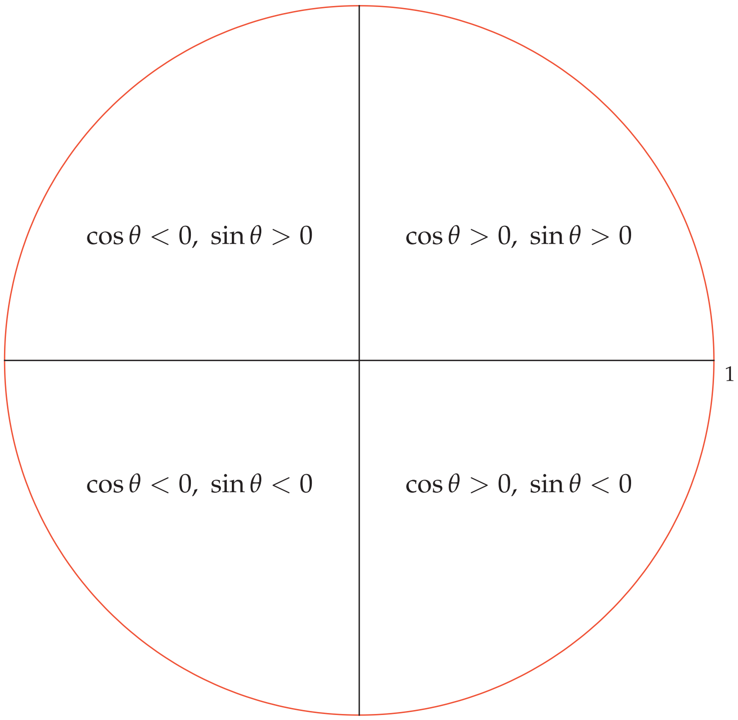
Note that cos and sin are functions; thus $\cos(\theta)$ and $\sin(\theta)$ might be a better notation than $\cos\theta$ and $\sin\theta$. In an expression such as

$$\frac{\cos 10}{\cos 5},$$

we cannot cancel cos in the numerator and denominator, just as we cannot cancel a function f in the numerator and denominator of $\frac{f(10)}{f(5)}$. Similarly, the expression above is not equal to $\cos 2$, just as $\frac{f(10)}{f(5)}$ is usually not equal to $f(2)$.

The Signs of Cosine and Sine

The coordinate axes divide the coordinate plane into four regions, often called **quadrants**. The quadrant in which a radius lies determines whether the cosine and sine of the corresponding angle are positive or negative. The figure below shows the sign of the cosine and the sign of the sine in each of the four quadrants. Thus, for example, an angle corresponding to a radius lying in the region marked "$\cos\theta < 0, \sin\theta > 0$" (the upper-left quadrant) will have a cosine that is negative and a sine that is positive.



There is no need to memorize this figure, because you can always reconstruct it if you understand the definitions of cosine and sine.

Recall that the cosine of an angle is the first coordinate of the endpoint of the corresponding radius. Thus the cosine is positive in the two quadrants where the first coordinate is positive, as shown in the figure above. Also, the cosine is negative in the two quadrants where the first coordinate is negative.

Similarly, the sine of an angle is the second coordinate of the endpoint of the corresponding radius. Thus the sine is positive in the two quadrants where the second coordinate is positive, as shown in the figure above. Also, the sine is negative in the two quadrants where the second coordinate is negative.

The next example should help you understand how the quadrant determines the sign of the cosine and sine.

To remember that $\cos\theta$ is the first coordinate of the endpoint of the radius corresponding to θ and $\sin\theta$ is the second coordinate, keep cosine *and* sine *in alphabetical order.*

Example 3

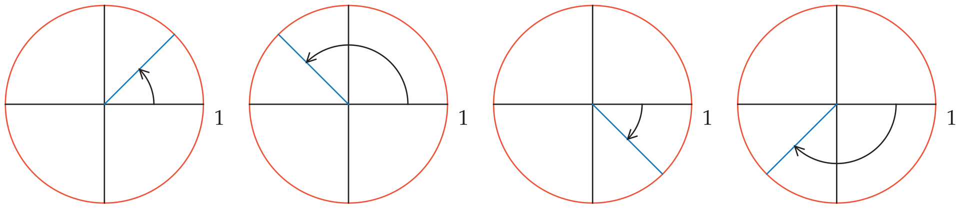
(a) Evaluate $\cos\frac{\pi}{4}$ and $\sin\frac{\pi}{4}$.

(b) Evaluate $\cos\frac{3\pi}{4}$ and $\sin\frac{3\pi}{4}$.

(c) Evaluate $\cos\left(-\frac{\pi}{4}\right)$ and $\sin\left(-\frac{\pi}{4}\right)$.

(d) Evaluate $\cos\left(-\frac{3\pi}{4}\right)$ and $\sin\left(-\frac{3\pi}{4}\right)$.

solution The four angles $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $-\frac{\pi}{4}$, and $-\frac{3\pi}{4}$ radians (or, equivalently, 45°, 135°, −45°, and −135°) are shown in the next figure. Each coordinate of the radius corresponding to each of these angles is either $\frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$; the only issue to worry about in computing the cosine and sine of these angles is the sign.



$\frac{\pi}{4}$ radians *$\frac{3\pi}{4}$ radians* *$-\frac{\pi}{4}$ radians* *$-\frac{3\pi}{4}$ radians*

(a) Both coordinates of the endpoint of the radius corresponding to $\frac{\pi}{4}$ radians are positive. Thus $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

Quadrants can be labeled by the descriptive terms upper/lower and right/left. For example, the radius corresponding to $\frac{3\pi}{4}$ radians (which equals $135°$) is in the upper-left quadrant.

(b) The first coordinate of the endpoint of the radius corresponding to $\frac{3\pi}{4}$ radians is negative; the second coordinate is positive. Thus $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$ and $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$.

(c) The first coordinate of the endpoint of the radius corresponding to $-\frac{\pi}{4}$ radians is positive and the second coordinate is negative. Thus $\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

(d) Both coordinates of the endpoint of the radius corresponding to $-\frac{3\pi}{4}$ radians are negative. Thus $\cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

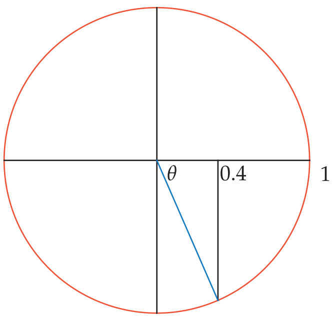
The next example shows how to use information about the signs of the cosine and sine to locate the corresponding radius.

Example 4

Sketch the radius of the unit circle corresponding to an angle θ such that

$$\cos\theta = 0.4 \quad \text{and} \quad \sin\theta < 0.$$

solution Because $\cos\theta$ is positive and $\sin\theta$ is negative, the radius corresponding to θ lies in the lower-right quadrant. To find the endpoint of this radius, which has first coordinate 0.4, start with the point 0.4 on the horizontal axis and then move vertically down to reach a point on the unit circle. Then draw the radius from the origin to that point, as shown here.



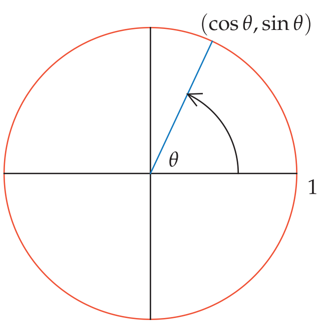
The radius corresponding to angle θ with $\cos\theta = 0.4$ and $\sin\theta < 0$.

The Key Equation Connecting Cosine and Sine

By definition of cosine and sine, the point $(\cos\theta, \sin\theta)$ is on the unit circle, which is the set of points in the coordinate plane such that the sum of the squares of the coordinates equals 1. In the xy-plane, the unit circle is described by the equation

$$x^2 + y^2 = 1.$$

Thus the following crucial equation holds.



The point $(\cos\theta, \sin\theta)$ is on the unit circle.

Relationship between cosine and sine

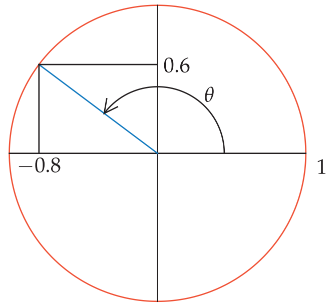
$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

for every angle θ.

Given either $\cos\theta$ or $\sin\theta$, the last equation can be used to solve for the other quantity, provided that we have enough additional information to determine the sign. The following example illustrates this procedure.

Example 5

Suppose θ is an angle such that $\sin\theta = 0.6$, and suppose also that $\frac{\pi}{2} < \theta < \pi$. Evaluate $\cos\theta$.



solution The equation above implies that $(\cos\theta)^2 + (0.6)^2 = 1$. Because $(0.6)^2 = 0.36$, this implies that

$$(\cos\theta)^2 = 0.64.$$

Thus $\cos\theta = 0.8$ or $\cos\theta = -0.8$. The additional information that $\frac{\pi}{2} < \theta < \pi$ implies that $\cos\theta$ is negative, as can been seen in the figure. Thus

$$\cos\theta = -0.8.$$

The Graphs of Cosine and Sine

Before graphing the cosine and sine functions, we should think carefully about the domain and range of these functions. Recall that for each real number θ, there is a radius of the unit circle corresponding to θ.

Recall also that the coordinates of the endpoints of the radius corresponding to the angle θ are labeled $(\cos\theta, \sin\theta)$, thus defining the cosine and sine functions. These functions are defined for every real number θ. Thus the domain of both cosine and sine is the set of real numbers.

As we have already noted, a consequence of $(\cos\theta, \sin\theta)$ lying on the unit circle is the equation

$$(\cos\theta)^2 + (\sin\theta)^2 = 1.$$

Because $(\cos\theta)^2$ and $(\sin\theta)^2$ are both nonnegative, the equation above implies that

$$(\cos\theta)^2 \le 1 \quad \text{and} \quad (\sin\theta)^2 \le 1.$$

Thus $\cos\theta$ and $\sin\theta$ must both be between -1 and 1.

Cosine and sine are between -1 and 1

$$-1 \le \cos\theta \le 1 \quad \text{and} \quad -1 \le \sin\theta \le 1$$

for every angle θ.

These inequalities can be used as a crude test of the plausibility of a result. For example, suppose you do a calculation and determine that $\cos\theta = 2$, which is impossible. Thus either there is a mistake in your calculation or solutions corresponding to $\cos\theta = 2$ should be discarded (see the solution to Exercise 33 for an example).

These inequalities could also be written in the following form:

$$|\cos\theta| \le 1 \quad \text{and} \quad |\sin\theta| \le 1.$$

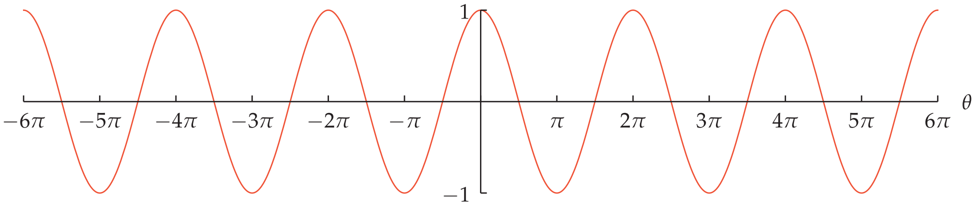
The first coordinates of the points of the unit circle are precisely the values of the cosine function. Every number in the interval $[-1, 1]$ is the first coordinate of some point on the unit circle. Thus we can conclude that the range of the cosine function is the interval $[-1, 1]$. A similar conclusion holds for the sine function (use second coordinates instead of first coordinates).

In summary, we have the following results.

Domain and range of cosine and sine

- The domain of both cosine and sine is the set of real numbers.
- The range of both cosine and sine is the interval $[-1, 1]$.

Because the domain of the cosine and the sine is the set of real numbers, we cannot show the graph of these functions on their entire domain. To understand what the graphs of these functions look like, we start by looking at the graph of cosine on the interval $[-6\pi, 6\pi]$.



The graph of cosine on the interval $[-6\pi, 6\pi]$.

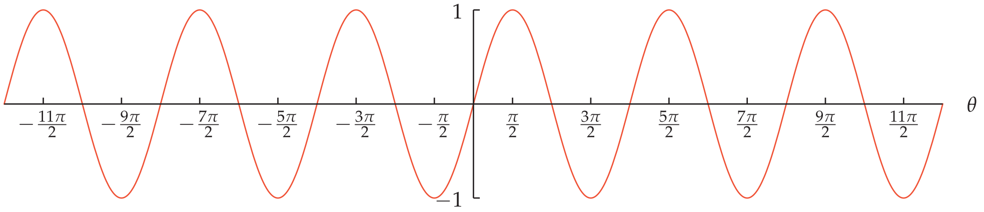
The Greek mathematician Hipparchus, depicted here in a 19^{th}*-century illustration, developed trigonometry over 2100 years ago as a tool for calculations in astronomy.*

Let's begin examining the graph above by noting that the point $(0, 1)$ is on the graph, as expected from the equation $\cos 0 = 1$. Note that the horizontal axis has been called the θ-axis.

Moving to the right along the θ-axis from the origin, we see that the graph crosses the θ-axis at the point $(\frac{\pi}{2}, 0)$, as expected from the equation $\cos \frac{\pi}{2} = 0$. Continuing further to the right, we see that the graph hits its lowest value when $\theta = \pi$, as expected from the equation $\cos \pi = -1$. The graph then crosses the θ-axis again at the point $(\frac{3\pi}{2}, 0)$, as expected from the equation $\cos \frac{3\pi}{2} = 0$. Then the graph hits its highest value again when $\theta = 2\pi$, as expected from the equation $\cos(2\pi) = 1$.

The most striking feature of the graph above is its periodic nature—the graph repeats itself. To understand why the graph of cosine exhibits this periodic behavior, consider a radius of the unit circle starting along the positive horizontal axis and moving counterclockwise. As the radius moves, the first coordinate of its endpoint gives the value of the cosine of the corresponding angle. After the radius moves through an angle of 2π radians, it returns to its original position. Then it begins the cycle again, returning to its original position after moving through a total angle of 4π, and so on. Thus we see the periodic behavior of the graph of cosine.

In Section 4.6 we will examine the properties of cosine and its graph more deeply. For now, let's turn to the graph of sine. Here is the graph of sine on the interval $[-6\pi, 6\pi]$.



The graph of sine on the interval $[-6\pi, 6\pi]$.

The word sine *comes from the Latin word* sinus, *which means curve.*

This graph goes through the origin, as expected because $\sin 0 = 0$. Moving to the right along the θ-axis from the origin, we see that the graph hits its highest value when $\theta = \frac{\pi}{2}$, as expected because $\sin \frac{\pi}{2} = 1$. Continuing further to the right, we see that the graph crosses the θ-axis at the point $(\pi, 0)$, as expected because $\sin \pi = 0$. The graph then hits its lowest value when $\theta = \frac{3\pi}{2}$, as expected because $\sin \frac{3\pi}{2} = -1$. Then the graph crosses the θ-axis again at $(2\pi, 0)$, as expected because $\sin(2\pi) = 0$.

Surely you have noticed that the graph of sine looks much like the graph of cosine. It appears that shifting one graph somewhat to the left or right produces the other graph. We will see that this is indeed the case when we delve more deeply into properties of cosine and sine in Section 4.6.

Exercises

Give exact values for the quantities in Exercises 1–10. Do not use a calculator for any of these exercises—otherwise you will likely get decimal approximations for some solutions rather than exact answers. More importantly, good understanding will come from working these exercises by hand.

1 (a) $\cos(3\pi)$ (b) $\sin(3\pi)$

2 (a) $\cos\left(-\frac{3\pi}{2}\right)$ (b) $\sin\left(-\frac{3\pi}{2}\right)$

3 (a) $\cos\frac{11\pi}{4}$ (b) $\sin\frac{11\pi}{4}$

4 (a) $\cos\frac{15\pi}{4}$ (b) $\sin\frac{15\pi}{4}$

5 (a) $\cos\frac{2\pi}{3}$ (b) $\sin\frac{2\pi}{3}$

6 (a) $\cos\frac{4\pi}{3}$ (b) $\sin\frac{4\pi}{3}$

7 (a) $\cos 210°$ (b) $\sin 210°$

8 (a) $\cos 300°$ (b) $\sin 300°$

9 (a) $\cos 360045°$ (b) $\sin 360045°$

10 (a) $\cos(-360030°)$ (b) $\sin(-360030°)$

11 Find the smallest number θ larger than 4π such that $\cos\theta = 0$.

12 Find the smallest number θ larger than 6π such that $\sin\theta = \frac{\sqrt{2}}{2}$.

13 Find the four smallest positive numbers θ such that $\cos\theta = 0$.

14 Find the four smallest positive numbers θ such that $\sin\theta = 0$.

15 Find the four smallest positive numbers θ such that $\sin\theta = 1$.

16 Find the four smallest positive numbers θ such that $\cos\theta = 1$.

17 Find the four smallest positive numbers θ such that $\cos\theta = -1$.

18 Find the four smallest positive numbers θ such that $\sin\theta = -1$.

19 Find the four smallest positive numbers θ such that $\sin\theta = \frac{1}{2}$.

20 Find the four smallest positive numbers θ such that $\cos\theta = \frac{1}{2}$.

21 Suppose $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \frac{2}{5}$. Evaluate $\sin\theta$.

22 Suppose $0 < \theta < \frac{\pi}{2}$ and $\sin\theta = \frac{3}{7}$. Evaluate $\cos\theta$.

23 Suppose $\frac{\pi}{2} < \theta < \pi$ and $\sin\theta = \frac{2}{9}$. Evaluate $\cos\theta$.

24 Suppose $\frac{\pi}{2} < \theta < \pi$ and $\sin\theta = \frac{3}{8}$. Evaluate $\cos\theta$.

25 Suppose $-\frac{\pi}{2} < \theta < 0$ and $\cos\theta = 0.1$. Evaluate $\sin\theta$.

26 Suppose $-\frac{\pi}{2} < \theta < 0$ and $\cos\theta = 0.3$. Evaluate $\sin\theta$.

27 Find the smallest number x such that

$$\sin(e^x) = 0.$$

28 Find the smallest number x such that

$$\cos(e^x + 1) = 0.$$

29 Find the smallest positive number x such that

$$\sin(x^2 + x + 4) = 0.$$

30 Find the smallest positive number x such that

$$\cos(x^2 + 2x + 6) = 0.$$

31 Let θ be the acute angle between the positive horizontal axis and the line with slope 3 through the origin. Evaluate $\cos\theta$ and $\sin\theta$.

32 Let θ be the acute angle between the positive horizontal axis and the line with slope 4 through the origin. Evaluate $\cos\theta$ and $\sin\theta$.

33 Suppose $5(\cos\theta)^2 - 7\cos\theta - 6 = 0$. Evaluate $|\sin\theta|$.

34 Suppose $13(\sin x)^2 - 14\sin x - 24 = 0$. Evaluate $|\cos x|$.

Problems

35 (a) Sketch a radius of the unit circle corresponding to an angle θ such that $\cos\theta = \frac{6}{7}$.

(b) Sketch another radius, different from the one in part (a), also illustrating $\cos\theta = \frac{6}{7}$.

36 (a) Sketch a radius of the unit circle corresponding to an angle θ such that $\sin\theta = -0.8$.

(b) Sketch another radius, different from the one in part (a), also illustrating $\sin\theta = -0.8$.

37 Find angles u and v such that $\cos u = \cos v$ but $\sin u \neq \sin v$.

38 Find angles u and v such that $\sin u = \sin v$ but $\cos u \neq \cos v$.

39 Show that $\ln(\cos\theta)$ is the average of $\ln(1 - \sin\theta)$ and $\ln(1 + \sin\theta)$ for every θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

40 Suppose you have borrowed two calculators from friends, but you do not know whether they are set to work in radians or degrees. Thus you ask each calculator to evaluate $\cos 3.14$. One calculator gives an answer of -0.999999; the other calculator gives an answer of 0.998499. Without further use of a calculator, how would you decide which calculator is using radians and which calculator is using degrees? Explain your answer.

41 Suppose you have borrowed two calculators from friends, but you do not know whether they are set to work in radians or degrees. Thus you ask each calculator to evaluate $\sin 1$. One calculator gives an answer of 0.017452; the other calculator gives an answer of 0.841471. Without further use of a calculator, how would you decide which calculator is using radians and which calculator is using degrees? Explain your answer.

42 A good scientific calculator will show that

$$\cos 710 \approx 0.999999998,$$

where of course the left side means the cosine of 710 radians. Thus $\cos 710$ is remarkably close to 1. Use the approximation $\pi \approx \frac{355}{113}$ (which has an error of less than 3×10^{-7}) to explain why $\cos 710 \approx 1$.

43 Suppose m is a real number. Let θ be the acute angle between the positive horizontal axis and the line with slope m through the origin. Evaluate $\cos\theta$ and $\sin\theta$.

44 Explain why there does not exist a real number x such that $2^{\sin x} = \frac{3}{7}$.

45 Explain why $\pi^{\cos x} < 4$ for every real number x.

46 Explain why $\frac{1}{3} < e^{\sin x}$ for every number real number x.

47 Explain why the equation

$$(\sin x)^2 - 4\sin x + 4 = 0$$

has no solutions.

48 Explain why the equation

$$(\cos x)^{99} + 4\cos x - 6 = 0$$

has no solutions.

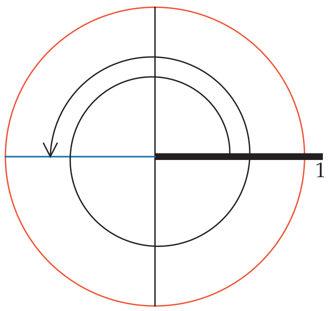
49 Explain why there does not exist a number θ such that $\log\cos\theta = 0.1$.

Worked-Out Solutions to Odd-Numbered Exercises

Give exact values for the quantities in Exercises 1–10. Do not use a calculator for any of these exercises—otherwise you will likely get decimal approximations for some solutions rather than exact answers. More importantly, good understanding will come from working these exercises by hand.

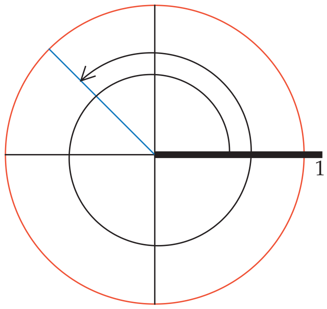
1 (a) $\cos(3\pi)$ (b) $\sin(3\pi)$

solution Because $3\pi = 2\pi + \pi$, an angle of 3π radians (as measured counterclockwise from the positive horizontal axis) consists of a complete revolution around the circle (2π radians) followed by another π radians (180°), as shown below. The endpoint of the corresponding radius is $(-1, 0)$. Thus $\cos(3\pi) = -1$ and $\sin(3\pi) = 0$.



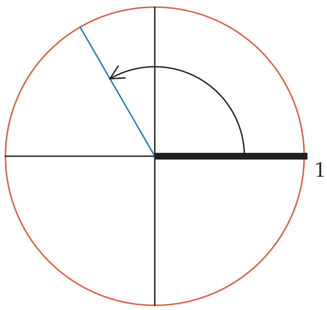
3 (a) $\cos\frac{11\pi}{4}$ (b) $\sin\frac{11\pi}{4}$

solution Because $\frac{11\pi}{4} = 2\pi + \frac{\pi}{2} + \frac{\pi}{4}$, an angle of $\frac{11\pi}{4}$ radians (as measured counterclockwise from the positive horizontal axis) consists of a complete revolution around the circle (2π radians) followed by another $\frac{\pi}{2}$ radians (90°), followed by another $\frac{\pi}{4}$ radians (45°), as shown below. Hence the endpoint of the corresponding radius is $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Thus $\cos\frac{11\pi}{4} = -\frac{\sqrt{2}}{2}$ and $\sin\frac{11\pi}{4} = \frac{\sqrt{2}}{2}$.



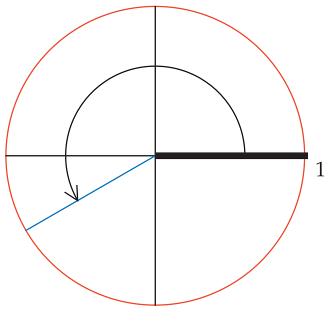
5 (a) $\cos\frac{2\pi}{3}$ (b) $\sin\frac{2\pi}{3}$

solution Because $\frac{2\pi}{3} = \frac{\pi}{2} + \frac{\pi}{6}$, an angle of $\frac{2\pi}{3}$ radians (as measured counterclockwise from the positive horizontal axis) consists of $\frac{\pi}{2}$ radians (90°) followed by another $\frac{\pi}{6}$ radians (30°), as shown below. The endpoint of the corresponding radius is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Thus $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.



7 (a) $\cos 210°$ (b) $\sin 210°$

solution Because $210 = 180 + 30$, an angle of 210° (as measured counterclockwise from the positive horizontal axis) consists of 180° followed by another 30°, as shown below. The endpoint of the corresponding radius is $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. Thus $\cos 210° = -\frac{\sqrt{3}}{2}$ and $\sin 210° = -\frac{1}{2}$.



9 (a) $\cos 360045°$ (b) $\sin 360045°$

solution Because $360045 = 360 \times 1000 + 45$, an angle of $360045°$ (as measured counterclockwise from the positive horizontal axis) consists of 1000 complete revolutions around the circle followed by another $45°$. The endpoint of the corresponding radius is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Thus

$$\cos 360045° = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin 360045° = \frac{\sqrt{2}}{2}.$$

11 Find the smallest number θ larger than 4π such that $\cos\theta = 0$.

solution Note that

$$0 = \cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = \cos\frac{5\pi}{2} = \dots$$

and that the only numbers whose cosine equals 0 are of the form $\frac{(2n+1)\pi}{2}$, where n is an integer. The smallest number of this form larger than 4π is $\frac{9\pi}{2}$. Thus $\frac{9\pi}{2}$ is the smallest number larger than 4π whose cosine equals 0.

13 Find the four smallest positive numbers θ such that $\cos\theta = 0$.

solution Think of a radius of the unit circle whose endpoint is $(1, 0)$. If this radius moves counterclockwise, forming an angle of θ radians with the positive horizontal axis, the first coordinate of its endpoint first becomes 0 when θ equals $\frac{\pi}{2}$ (which equals $90°$), then again when θ equals $\frac{3\pi}{2}$ (which equals $270°$), then again when θ equals $\frac{5\pi}{2}$ (which equals $360° + 90°$, or $450°$), then again when θ equals $\frac{7\pi}{2}$ (which equals $360° + 270°$, or $630°$), and so on. Thus the four smallest positive numbers θ such that $\cos\theta = 0$ are $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, and $\frac{7\pi}{2}$.

15 Find the four smallest positive numbers θ such that $\sin\theta = 1$.

solution Think of a radius of the unit circle whose endpoint is $(1, 0)$. If this radius moves counterclockwise, forming an angle of θ radians with the positive horizontal axis, then the second coordinate of its endpoint first becomes 1 when θ equals $\frac{\pi}{2}$ (which equals $90°$), then again when θ equals $\frac{5\pi}{2}$ (which equals $360° + 90°$, or $450°$), then again when θ equals $\frac{9\pi}{2}$ (which equals $2 \times 360° + 90°$, or $810°$), then again when θ equals $\frac{13\pi}{2}$ (which equals $3 \times 360° + 90°$, or $1170°$), and so on. Thus the four smallest positive numbers θ such that $\sin\theta = 1$ are $\frac{\pi}{2}$, $\frac{5\pi}{2}$, $\frac{9\pi}{2}$, and $\frac{13\pi}{2}$.

17 Find the four smallest positive numbers θ such that $\cos\theta = -1$.

solution Think of a radius of the unit circle whose endpoint is $(1, 0)$. If this radius moves counterclockwise, forming an angle of θ radians with the positive horizontal axis, the first coordinate of its endpoint first becomes -1 when θ equals π (which equals $180°$), then again when θ equals 3π (which equals $360° + 180°$, or $540°$), then again when θ equals 5π (which equals $2 \times 360° + 180°$, or $900°$), then again when θ equals 7π (which equals $3 \times 360° + 180°$, or $1260°$), and so on. Thus the four smallest positive numbers θ such that $\cos\theta = -1$ are π, 3π, 5π, and 7π.

19 Find the four smallest positive numbers θ such that $\sin\theta = \frac{1}{2}$.

solution Think of a radius of the unit circle whose endpoint is $(1, 0)$. If this radius moves counterclockwise, forming an angle of θ radians with the positive horizontal axis, the second coordinate of its endpoint first becomes $\frac{1}{2}$ when θ equals $\frac{\pi}{6}$ (which equals $30°$), then again when θ equals $\frac{5\pi}{6}$ (which equals $150°$), then again when θ equals $\frac{13\pi}{6}$ (which equals $360° + 30°$, or $390°$), then again when θ equals $\frac{17\pi}{6}$ (which equals $360° + 150°$, or $510°$), and so on. Thus the four smallest positive numbers θ such that $\sin\theta = \frac{1}{2}$ are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{13\pi}{6}$, and $\frac{17\pi}{6}$.

21 Suppose $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \frac{2}{5}$. Evaluate $\sin\theta$.

solution We know that

$$(\cos\theta)^2 + (\sin\theta)^2 = 1.$$

Thus

$$\begin{aligned}(\sin\theta)^2 &= 1 - (\cos\theta)^2\\ &= 1 - \left(\tfrac{2}{5}\right)^2\\ &= \frac{21}{25}.\end{aligned}$$

Because $0 < \theta < \frac{\pi}{2}$, we know that $\sin\theta > 0$. Thus taking square roots of both sides of the equation above gives

$$\sin\theta = \frac{\sqrt{21}}{5}.$$

23 Suppose $\frac{\pi}{2} < \theta < \pi$ and $\sin\theta = \frac{2}{9}$. Evaluate $\cos\theta$.

solution We know that

$$(\cos\theta)^2 + (\sin\theta)^2 = 1.$$

Thus

$$\begin{aligned}(\cos\theta)^2 &= 1-(\sin\theta)^2\\ &= 1-\left(\tfrac{2}{9}\right)^2\\ &= \tfrac{77}{81}.\end{aligned}$$

Because $\frac{\pi}{2} < \theta < \pi$, we know that $\cos\theta < 0$. Thus taking square roots of both sides of the equation above gives

$$\cos\theta = -\frac{\sqrt{77}}{9}.$$

25 Suppose $-\frac{\pi}{2} < \theta < 0$ and $\cos\theta = 0.1$. Evaluate $\sin\theta$.

solution We know that

$$(\cos\theta)^2 + (\sin\theta)^2 = 1.$$

Thus

$$\begin{aligned}(\sin\theta)^2 &= 1-(\cos\theta)^2\\ &= 1-(0.1)^2\\ &= 0.99.\end{aligned}$$

Because $-\frac{\pi}{2} < \theta < 0$, we know that $\sin\theta < 0$. Thus taking square roots of both sides of the equation above gives

$$\sin\theta = -\sqrt{0.99} \approx -0.995.$$

27 Find the smallest number x such that

$$\sin(e^x) = 0.$$

solution Note that e^x is an increasing function. Because e^x is positive for every real number x, and because π is the smallest positive number whose sine equals 0, we want to choose x so that $e^x = \pi$. Thus $x = \ln\pi$.

29 Find the smallest positive number x such that

$$\sin(x^2 + x + 4) = 0.$$

solution Note that $x^2 + x + 4$ is an increasing function on the interval $[0, \infty)$. If x is positive, then $x^2 + x + 4 > 4$. Because 4 is larger than π but less than 2π, the smallest number bigger than 4 whose sine equals 0 is 2π. Thus we want to choose x so that $x^2 + x + 4 = 2\pi$. In other words, we need to solve the equation

$$x^2 + x + (4 - 2\pi) = 0.$$

Using the quadratic formula, we see that the solutions to this equation are

$$x = \frac{-1 \pm \sqrt{8\pi - 15}}{2}.$$

A calculator shows that choosing the plus sign in the equation above gives $x \approx 1.0916$ and choosing the minus sign gives $x \approx -2.0916$. We seek only positive values of x, and thus we choose the plus sign in the equation above, getting $x \approx 1.0916$.

31 Let θ be the acute angle between the positive horizontal axis and the line with slope 3 through the origin. Evaluate $\cos\theta$ and $\sin\theta$.

solution From the solution to Exercise 5 in Section 4.1, we see that the endpoint of the relevant radius on the unit circle has coordinates $\left(\frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10}\right)$. Thus

$$\cos\theta = \frac{\sqrt{10}}{10} \quad\text{and}\quad \sin\theta = \frac{3\sqrt{10}}{10}.$$

33 Suppose $5(\cos\theta)^2 - 7\cos\theta - 6 = 0$. Evaluate $|\sin\theta|$.

solution Let $y = \cos\theta$. The equation above can be rewritten as

$$5y^2 - 7y - 6 = 0.$$

Solving for y (using either the quadratic formula or by factoring the left side of the equation above), we have

$$y = -\frac{3}{5} \quad\text{or}\quad y = 2.$$

However, $y = \cos\theta$, and there does not exist a real number θ such that $\cos\theta = 2$. Hence $y = -\frac{3}{5}$, and hence $\cos\theta = -\frac{3}{5}$. Now

$$\begin{aligned}(\sin\theta)^2 &= 1-(\cos\theta)^2\\ &= 1-\frac{9}{25}\\ &= \frac{16}{25}.\end{aligned}$$

Thus $\sin\theta = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$. Thus $|\sin\theta| = \frac{4}{5}$.